COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review**. *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Suppression of chaos by resonant parametric perturbations"

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(Received 26 June 1992)

The inhibition of the chaotic behavior of the Duffing-Holmes oscillator by means of a small parametric perturbation of suitable frequency, as shown by Lima and Pettini [Phys. Rev. A **41**, 726 (1990)], is here reconsidered. The discrepancies noted in that article between the analytical (Melnikov method) and numerical (Lyapunov exponents) results are shown to be basically due to an error in the calculation of the Melnikov distance rather than simply to the method's perturbative nature.

PACS number(s): 05.45.+b, 05.40.+j

Motivated by investigation into the mechanisms that can reduce or suppress chaos in nonlinear systems (such as various types of oscillators), Lima and Pettini have recently considered [1] the Duffing-Holmes oscillator with a parametric perturbation of the cubic term:

$$\ddot{x}-x+\beta[1+\eta\cos(\Omega t)]x^3=-\delta\dot{x}+\gamma\cos(\omega t)\ .$$

Here Ω and η are the frequency and amplitude, respectively, of the parametric perturbation, $\eta \ll 1$. They apply Melnikov's method (MM) to get predictions in analytical form, finding that these contrast with the results of numerical methods—Lyapunov characteristic exponents (LE's), power spectrum, and correlation function of the solution—near conditions of resonance ($\Omega \simeq \Omega_R^{(K)} \equiv K\omega$, $K=1,2,\ldots$). In their conclusions, the authors observe that, close to the principal resonance, the numerical calculations of the LE, λ, confirm the analytical prediction of the suppression of chaos, adding textually: "This phenomenon occurs for values of the perturbation amplitude greater than 0.1 times theoretically estimated value. . . ." They also note that, for the second and third harmonics of the forcing frequency ω, the suppression of chaos in not predicted by MM, while the LE vanishes for these values. They attempt to explain these quantitative and qualitative discrepancies between theoretical and numerical predictions by attributing them to the perturbative nature of the MM.

We shall show in this Comment that the qualitative, and to a lesser degree, the quantitative discrepancies originate in an error in the calculation of the Melnikov distance. Thus, for clarity using the notation of Ref. [1], one has that the Melnikov distance for the Duffing-Holmes equation with parametric perturbation of the cubic term is

$$\Delta(t_0)=\pi\left[\frac{2}{\beta}\right]^{1/2}\gamma\omega\,\mathrm{sech}\left[\frac{\pi\omega}{2}\right]\sin(\omega t_0)-\frac{4\delta}{3\beta}$$
$$-\eta\beta\int_{-\infty}^{\infty}dt\,p_0(t-t_0)[x_0(t-t_0)]^3\cos(\Omega t)\ ,$$

where

$$x_0(t)=\left[\frac{2}{\beta}\right]^{1/2}\mathrm{sech}t\ ,$$

$$p_0(t)=-\left[\frac{2}{\beta}\right]^{1/2}\mathrm{sech}t\,\tanh t$$

are the parametric equations of the homoclinic loop corresponding to the hyperbolic fixed point of the Duffing-Holmes equation with $\eta=\delta=\gamma=0$. After some simple algebraic manipulation, the last term on the right-hand side of the Melnikov distance can be recast into the form

$$\frac{4\eta}{\beta}\sin(\Omega t_0)\int_{-\infty}^{\infty}d\tau\cosh^{-5}\tau\sinh\tau\sin(\Omega\tau)\ .$$

The last integral can be calculated by the method of residues [2]. Its value is $(\pi/24)(\Omega^4+4\Omega^2)\mathrm{csch}(\pi\Omega/2)$, contrasting with the value given in Ref. [1], $(\pi/24)(\Omega^4-6\Omega^2+1)\mathrm{csch}(\pi\Omega/2)$. Finally, after some obvious substitutions, one has for the Melnikov distance

$$\Delta(t_0)=A(\omega)\sin(\omega t_0)-B'(\Omega)\sin(\Omega t_0)-C\ ,$$

with

$$A(\omega)=\frac{\sqrt{2}}{\sqrt{\beta}}\pi\gamma\omega\,\mathrm{sech}\left[\frac{\pi\omega}{2}\right],$$

$$B'(\Omega)=\frac{\pi\eta}{6\beta}(\Omega^4+4\Omega^2)\mathrm{csch}\left[\frac{\pi\Omega}{2}\right],$$

$$C=\frac{4\delta}{3\beta},$$

whereas Eq. (12) of Ref. [1] is

$$\Delta(t_0)=2A(\omega)\sin(\omega t_0)+B(\Omega)\sin(\Omega t_0)+C,$$

with

$$B(\Omega)=\frac{\pi\eta}{6\beta}(\Omega^4-6\Omega^2+1)\mathrm{csch}\left[\frac{\pi\Omega}{2}\right].$$

Figures 2 and 3 of Ref. [1] show the reciprocal of the time τ_M elapsed between two homoclinic intersections as a function of Ω for different values of the parameters β, δ, γ, ω, and η. These graphs are affected by the error in $B(\Omega)$, since τ_M is calculated from the Melnikov distance.

When the corrections introduced are taken into account, Lemma 1 of Ref. [1] changes so that now the necessary and sufficient condition for $\Delta(t_0)$ to have always the same sign, i.e., $\Delta(t_0)<0$, when Ω is in resonance with the driving frequency ω, is written

$$\eta>\eta'_{\min}=\left|\frac{6\beta[A(\omega)-C]}{\pi(\Omega^4+4\Omega^2)\mathrm{csch}(\pi\Omega/2)}\right|,$$

instead of

$$\eta>\eta_{\min}=\left|\frac{6\beta[2A(\omega)-C]}{\pi(\Omega^4-6\Omega^2+1)\mathrm{csch}(\pi\Omega/2)}\right|,$$

proposed in Ref. [1]. Figure 1 shows a plot of the two functions $D(\Omega)=\eta_{\min}/|6\beta[2A(\omega)-C]|$ and $D'(\Omega)=\eta'_{\min}/|6\beta[A(\omega)-C]|$. Notice the radically different behavior of the two functions at $\Omega=0$, where $D(\Omega)=0$, while $\lim_{\Omega\to 0}D'(\Omega)=\infty$, and at $\Omega_1=0.4142$, $\Omega_2=2.4142$, where $D'(\Omega)$ remains finite and $\lim_{\Omega\to\Omega_{1,2}}D(\Omega)=\infty$. In particular, for $\Omega>1/\sqrt{10}$, $D'(\Omega)<D(\Omega)$. This implies that, for a given set of parameter values, $\eta'_{\min}<\eta_{\min}$, i.e., $\eta_{\min}$ overestimates the threshold for the suppression of chaos, as was mentioned in Ref. [1]. Another point is that the values of ω used throughout Ref. [1] to compare the analytical results with the numerical are $\omega_{\rm I}=1.1$ and $\omega_{\rm II}=1.22$. For these values, Fig. 1 shows that, near the principal resonance $\Omega\simeq\Omega_{\rm I}^{(1)},\Omega_{\rm II}^{(1)}$, the functions $D(\Omega)$ and $D'(\Omega)$ are very close together. This is why, despite the error introduced with $B(\Omega)$, Lima and Pettini were able to point to qualitative agreement between MM and LE for the suppression of chaos when the forcing frequency coincides with the frequency of the parametric excitation. It is also clear at first sight that, near the resonances corresponding to the second and third harmonics of the forcing frequency, $\Omega\simeq\Omega_{\rm I}^{(2)},\Omega_{\rm II}^{(2)}$, $\Omega\simeq\Omega_{\rm I}^{(3)},\Omega_{\rm II}^{(3)}$, the value of $D(\Omega)$ increases markedly as one is very close to $\Omega_2=2.4142$ for which $D(\Omega)\to\infty$, while $D'(\Omega)$ remains close to its minimum. Simple calculation, in particular for the parameter set employed in Ref. [1], shows that the degree of agreement between the MM and LE predictions is the same near the principal resonance as near the resonances with the second and third harmonics of the forcing frequency. For example, for the standard parameter set used in Ref. [1] ($\beta=4$, $\delta=0.154$, $\gamma=0.088$, $\omega=\omega_{\rm I}\equiv 1.1$), the MM yields $\eta'_{\min}=(0.0750, 0.0642, 0.0955)$ for $\Omega=(\omega_{\rm I},2\omega_{\rm I},3\omega_{\rm I})$, while the LE vanishes for $\eta'_{\min}=0.03$ [1]. These discrepancies subsist because of the perturbative (to first order) nature of the MM. Nonetheless, this method shows itself to be a powerful tool for predicting the suppression of homoclinic chaos.

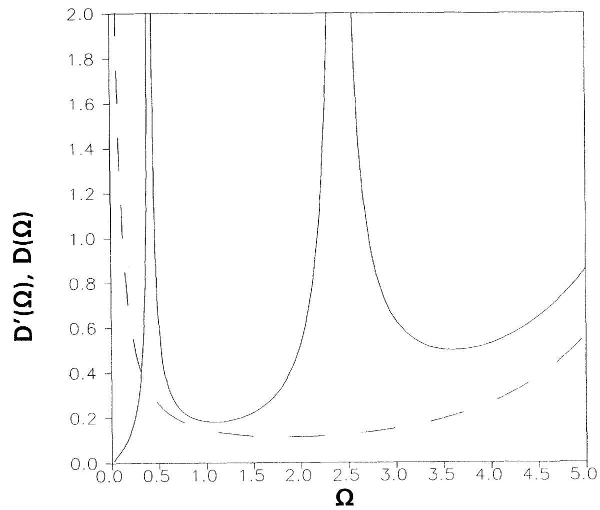

FIG. 1. Functions $D(\Omega)$ and $D'(\Omega)$. The solid line represents $D(\Omega)$ and the dashed line $D'(\Omega)$.

Finally, we should note that in a recent work by Fronzoni, Giocondo, and Pettini [3], the erroneous value of the Melnikov distance is again used as in Ref. [1], and the same conclusions are drawn that have been shown to be mistaken in this Comment.

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[1] R. Lima and M. Pettini, Phys. Rev. A **41**, 726 (1990).
[2] F. Cuadros and R. Chacón (unpublished).
[3] L. Fronzoni, M. Giocondo, and M. Pettini, Phys. Rev. A **43**, 6483 (1991).